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# Linear response for intermittent maps with summable and nonsummable decay of correlations

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## Abstract

We consider a family of Pomeau-Manneville type interval maps  $T_\alpha$ , parametrized by  $\alpha \in (0, 1)$ , with the unique absolutely continuous invariant probability measures  $\nu_\alpha$ , and rate of correlations decay  $n^{1-1/\alpha}$ . We show that despite the absence of a spectral gap for all  $\alpha \in (0, 1)$  and despite nonsummable correlations for  $\alpha \geq 1/2$ , the map  $\alpha \mapsto \int \varphi d\nu_\alpha$  is continuously differentiable for  $\varphi \in L^q[0, 1]$  for  $q$  sufficiently large.

## 1 Introduction

Let  $T_\alpha: X \rightarrow X$  be a family of transformations on a Riemannian manifold  $X$  parametrized by  $\alpha$  and admitting unique SRB measures  $\nu_\alpha$ . Having an observable  $\varphi: X \rightarrow \mathbb{R}$ , it may be important to know how  $\int \varphi d\nu_\alpha$  changes with  $\alpha$ . If the map  $\alpha \mapsto \int \varphi d\nu_\alpha$  is differentiable, then *linear response* holds.

An interesting question is, which families of maps and observables have linear response. Ruelle proved linear response in the Axiom A case [R97, R98, R09, R09.1]. It was shown in [D04, B07, M07, BS08] that spectral gap and structural stability are not necessary or sufficient conditions.

We consider a family of Pomeau-Manneville type maps with slow (polynomial) decay of correlations:  $T_\alpha: [0, 1] \rightarrow [0, 1]$ , given by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, 1/2] \\ 2x - 1 & \text{if } x \in (1/2, 1] \end{cases}, \quad (1.1)$$

parametrized by  $\alpha \in [0, 1)$ . By [LSV99], each  $T_\alpha$  admits a unique absolutely continuous invariant probability measure  $\nu_\alpha$ , and the sharp rate of decay of correlations for Hölder observables is  $n^{1-1/\alpha}$  [Y99, S02, G04, H04].

We prove linear response on the interval  $\alpha \in (0, 1)$ , including the case when  $\alpha \geq 1/2$ , and correlations are not summable. This is the first time that linear response has been proved in the case of nonsummable decay of correlations. We develop a machinery which, when applied to the family  $T_\alpha$ , yields:

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**Theorem 1.1.** *For any  $\varphi \in C^1[0, 1]$ , the map  $\alpha \mapsto \int \varphi d\nu_\alpha$  is continuously differentiable on  $(0, 1)$ .*

Using additional structure of the family  $T_\alpha$ , we prove a stronger result:

**Theorem 1.2.** *Let  $\rho_\alpha$  be the density of  $\nu_\alpha$ . For every  $\alpha, x \in (0, 1) \times (0, 1]$  there exists a partial derivative  $\partial_\alpha \rho_\alpha(x)$ . Both  $\rho_\alpha(x)$  and  $\partial_\alpha \rho_\alpha(x)$  are jointly continuous in  $\alpha, x$  on  $(0, 1) \times (0, 1]$ . Moreover, for every interval  $[\alpha_-, \alpha_+] \subset (0, 1)$  there exists a constant  $K$ , such that for all  $x \in (0, 1]$  and  $\alpha \in [\alpha_-, \alpha_+]$*

$$\rho_\alpha(x) \leq K x^{-\alpha} \quad \text{and} \quad |\partial_\alpha \rho_\alpha(x)| \leq K x^{-\alpha}(1 - \log x).$$

*In particular, for any  $q > (1 - \alpha_+)^{-1}$  and observable  $\varphi \in L^q[0, 1]$ , the map  $\alpha \mapsto \int \varphi d\nu_\alpha$  is continuously differentiable on  $[\alpha_-, \alpha_+]$ .*

The same problem of linear response for the family  $T_\alpha$  has been solved independently by Baladi and Todd [BT15] using different methods. They prove that for  $\alpha_+ \in (0, 1)$ ,  $q > (1 - \alpha_+)^{-1}$  and  $\varphi \in L^q[0, 1]$ , the map  $\alpha \mapsto \int \varphi d\nu_\alpha$  is differentiable on  $[0, \alpha_+)$ , plus they give an explicit formula for the derivative in terms of the transfer operator corresponding to  $T_\alpha$ . We obtain more control of the invariant measure, as in Theorem 1.2, but do not give such a formula. Instead we provide explicit formulas for  $\rho_\alpha$  and  $\partial_\alpha \rho_\alpha$  in terms of the transfer operator for the induced map (see Subsections 4.1 and 5.3), but we do not state them here because they are too technical. Whereas [BT15] were the first to treat the case  $\alpha < 1/2$ , we were the first to treat the case  $\alpha \geq 1/2$ .

In a more recent paper [BS15], Bahsoun and Saussol consider a class of dynamical systems which includes (1.1). They prove in particular that for  $\beta \in (0, 1)$  and  $\alpha \in (0, \beta)$ ,

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in (0, 1]} x^\beta \left| \frac{\rho_{\alpha+\varepsilon} - \rho_\alpha}{\varepsilon} - \partial_\alpha \rho_\alpha \right| = 0.$$

That is,  $\rho_\alpha$  is differentiable as an element of a Banach space of continuous functions on  $(0, 1]$  with a norm  $\|\varphi\| = \sup_{x \in (0, 1]} x^\beta |\varphi(x)|$ . We remark that for the map (1.1) this follows from Theorem 1.2.

The paper is organized as follows. In Section 2 we introduce an abstract framework, and in Section 3 we apply it to the family  $T_\alpha$ , and prove Theorem 1.1.

Technical parts of proofs are presented separately: in Section 4 for the abstract framework, and in Section 5 for the properties of the family  $T_\alpha$ .

Theorem 1.2 is proven in Subsection 5.3. We do not give the proof earlier in the paper, because it uses rather special technical properties of  $T_\alpha$ .

## 2 Setup and Notations

Let  $I \subset \mathbb{R}$  be a closed bounded interval, and  $F_\alpha : I \rightarrow I$  be a family of maps, parametrized by  $\alpha \in [\alpha_-, \alpha_+]$ . Assume that each  $F_\alpha$  has finitely or countably many full branches, indexed by  $r \in \mathcal{R}$ , the same set  $\mathcal{R}$  for all  $\alpha$ .

Technically we assume that  $I = \bigcup_r [a_r, b_r]$  modulo a zero measure set (branch boundaries  $a_r$  and  $b_r$  may depend on  $\alpha$ ), and that for each  $r$  the map  $F_{\alpha, r} : [a_r, b_r] \rightarrow I$  is a diffeomorphism; here  $F_{\alpha, r}$  equals to  $F_\alpha$  on  $(a_r, b_r)$ , and is extended continuously to  $[a_r, b_r]$ .

- We use the letter  $\xi$  for spatial variable, and notation  $(\cdot)'$  for differentiation with respect to  $\xi$ , and  $\partial_\alpha$  for differentiation with respect to  $\alpha$ .
- Denote  $G_{\alpha,r} = |(F_{\alpha,r}^{-1})'|$ , defined on  $I$ . Note that  $G_{\alpha,r} = \pm(F_{\alpha,r}^{-1})'$ , the sign depends only on  $r$ .
- For each  $i$  let  $\|h\|_{C^i} = \max(\|h\|_\infty, \|h'\|_\infty, \dots, \|h^{(i)}\|_\infty)$  denote the  $C^i$  norm of  $h$ .
- Let  $m$  be the normalized Lebesgue measure on  $I$ , and  $P_\alpha$  be the transfer operator for  $F_\alpha$  with respect to  $m$ . By definition,  $\int (P_\alpha u)v \, dm = \int u(v \circ F_\alpha) \, dm$  for  $u \in L^1(I)$  and  $v \in L^\infty(I)$ . There is an explicit formula for  $P_\alpha$ :

$$(P_\alpha h)(\xi) = \sum_r G_{\alpha,r}(\xi) h(F_{\alpha,r}^{-1}(\xi)).$$

We assume that  $F_{\alpha,r}^{-1}$  and  $G_{\alpha,r}$ , as functions of  $\alpha$  and  $\xi$ , have continuous second order partial derivatives for each  $r \in \mathcal{R}$ , and there are constants

$$0 < \sigma < 1, \quad K_0 > 0 \quad \text{and} \quad \gamma_r \geq 1, \quad r \in \mathcal{R}$$

such that uniformly in  $\alpha \in [\alpha_-, \alpha_+]$  and  $r \in \mathcal{R}$ :

$$\begin{aligned} \text{A1. } \|G_{\alpha,r}\|_\infty &\leq \sigma, & \text{A4. } \|\partial_\alpha F_{\alpha,r}^{-1}\|_\infty &\leq \gamma_r, \\ \text{A2. } \|G'_{\alpha,r}/G_{\alpha,r}\|_\infty &\leq K_0, & \text{A5. } \|(\partial_\alpha G_{\alpha,r})/G_{\alpha,r}\|_\infty &\leq \gamma_r, \\ \text{A3. } \|G''_{\alpha,r}/G_{\alpha,r}\|_\infty &\leq K_0, & \text{A6. } \|(\partial_\alpha G'_{\alpha,r})/G_{\alpha,r}\|_\infty &\leq \gamma_r, \\ & & \text{A7. } \sum_r \|G_{\alpha,r}\|_\infty \gamma_r &\leq K_0. \end{aligned}$$

It is well known that under conditions A1, A2, the map  $F_\alpha$  admits a unique absolutely continuous invariant measure (see for example [P80, Z04]), which we denote by  $\mu_\alpha$ , and its density by  $h_\alpha = d\mu_\alpha/dm$ .

**Theorem 2.1.**  $h_\alpha \in C^2(I)$  and  $\partial_\alpha h_\alpha \in C^1(I)$  for each  $\alpha \in [\alpha_-, \alpha_+]$ . The maps

$$\begin{aligned} [\alpha_-, \alpha_+] &\longrightarrow C^2(I) & \text{and} & & [\alpha_-, \alpha_+] &\longrightarrow C^1(I) \\ \alpha &\longmapsto h_\alpha & & & \alpha &\longmapsto \partial_\alpha h_\alpha \end{aligned}$$

are continuous.

The proof of Theorem 2.1 is postponed until Section 4.

*Remark 2.2.* Later in the proof of Theorem 2.1 we explicitly compute constants  $K_1$  and  $K_2$ , such that  $\|h_\alpha\|_{C^2} \leq K_1$  and  $\|\partial_\alpha h_\alpha\|_{C^1} \leq K_2$ . These constants depend only on  $K_0$  and  $\sigma$ . Below we use  $K_1$  and  $K_2$  as reference bounds on  $\|h_\alpha\|_{C^2}$  and  $\|\partial_\alpha h_\alpha\|_{C^1}$ .

*Remark 2.3.* Both  $h_\alpha(\xi)$  and  $(\partial_\alpha h_\alpha)(\xi)$  are continuous in  $\alpha$ , and continuous in  $\xi$  uniformly in  $\alpha$ , because  $\|h_\alpha\|_{C^2} \leq K_1$ . Therefore both are jointly continuous in  $\alpha$  and  $\xi$ .

**Corollary 2.4.** Assume that  $\Phi_\alpha$  is a family of observables, such that  $\Phi_\alpha(F_{\alpha,r}^{-1}(\xi))$  and  $\partial_\alpha[\Phi_\alpha(F_{\alpha,r}^{-1}(\xi))]$  are jointly continuous in  $\alpha$  and  $\xi$  for each  $r$ , and

$$\|\Phi_\alpha \circ F_{\alpha,r}^{-1}\|_\infty \leq \delta_r, \quad \|\partial_\alpha[\Phi_\alpha \circ F_{\alpha,r}^{-1}]\|_\infty \leq \delta_r$$

for some constants  $\delta_r \geq 1$ ,  $r \in \mathcal{R}$ . Assume also that  $\sum_r \gamma_r \delta_r \|G_{\alpha,r}\|_\infty \leq K_3$ . Then the map  $\alpha \mapsto \int \Phi_\alpha d\mu_\alpha$  is continuously differentiable on  $[\alpha_-, \alpha_+]$ .

*Proof.* First,

$$\begin{aligned} \int \Phi_\alpha d\mu_\alpha &= \int h_\alpha \Phi_\alpha dm = \int P_\alpha(h_\alpha \Phi_\alpha) dm \\ &= \int \left( \sum_r (h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r} \right) dm. \end{aligned}$$

By Theorem 2.1 and our assumptions,  $h_\alpha \circ F_{\alpha,r}^{-1}$ ,  $\Phi_\alpha \circ F_{\alpha,r}^{-1}$  and  $G_{\alpha,r}$  are jointly continuous in  $\alpha$  and  $\xi$ , and have jointly continuous partial derivatives by  $\alpha$ . Since  $\|h_\alpha\|_{C^2} \leq K_1$ ,  $\|\Phi_\alpha \circ F_{\alpha,r}^{-1}\|_\infty \leq \delta_r$  and  $\sum_r \delta_r \gamma_r \|G_{\alpha,r}\|_\infty \leq K_3$ , the series inside the integral converges uniformly to a function which is jointly continuous in  $\alpha$  and  $\xi$ . Therefore,  $\int \Phi_\alpha d\mu_\alpha$  depends continuously on  $\alpha$ .

Moreover, since  $\|\partial_\alpha h_\alpha\|_{C^1} \leq K_2$ ,  $\|\partial_\alpha F_{\alpha,r}^{-1}\|_\infty \leq \gamma_r$ ,  $|\partial_\alpha(\Phi_\alpha \circ F_{\alpha,r}^{-1})| \leq \delta_r$ ,  $|\partial_\alpha(\Phi_\alpha \circ F_{\alpha,r}^{-1})| \leq \delta_r$  and  $|\partial_\alpha[G_{\alpha,r}]| \leq \gamma_r G_{\alpha,r}$ , we can write

$$\begin{aligned} |\partial_\alpha(h_\alpha \circ F_{\alpha,r}^{-1})| &= |[\partial_\alpha h_\alpha] \circ F_{\alpha,r}^{-1} + (h'_\alpha \circ F_{\alpha,r}^{-1}) \partial_\alpha F_{\alpha,r}^{-1}| \\ &\leq K_2 + K_1 \gamma_r \quad \text{and} \\ |\partial_\alpha[(h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r}]| &\leq [(K_2 + K_1 \gamma_r) + K_1 + K_1 \gamma_r] \delta_r G_{\alpha,r} \\ &\leq (3K_1 + K_2) \delta_r \gamma_r G_{\alpha,r}. \end{aligned}$$

Since  $\sum_r \delta_r \gamma_r \|G_{\alpha,r}\|_\infty \leq K_3$ , we can write

$$\begin{aligned} \frac{d}{d\alpha} \int \Phi_\alpha d\mu_\alpha &= \frac{d}{d\alpha} \int \left( \sum_r (h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r} \right) dm \\ &= \sum_r \int \partial_\alpha[(h_\alpha \circ F_{\alpha,r}^{-1})(\Phi_\alpha \circ F_{\alpha,r}^{-1}) G_{\alpha,r}] dm. \end{aligned}$$

The series converges uniformly, and is bounded by  $K_3(3K_1 + K_2)$ . The terms are continuous in  $\alpha$ , thus so is the sum.  $\square$

### 3 Application to Pomeau-Manneville type maps

In this section we work with the family of maps  $T_\alpha$ , defined by equation (1.1). Assume that  $\alpha \in [\alpha_-, \alpha_+] \subset (0, 1)$ . Let  $\tau_\alpha(x) = \min\{k \geq 1: T_\alpha^k x \in [1/2, 1]\}$  be the return time to the interval  $[1/2, 1]$ . Let

$$F_\alpha: [1/2, 1] \rightarrow [1/2, 1], \quad x \mapsto T_\alpha^{\tau_\alpha(x)}(x)$$

be the induced map.

**Branches.** Let  $x_0 = 1$ ,  $x_1 = 1/2$ , and define  $x_k \in (0, 1/2]$  for  $k \geq 1$  by setting  $x_k = T_\alpha x_{k+1}$ . Note that  $T_\alpha^k: (x_{k+1}, x_k) \rightarrow (1/2, 1)$  is a diffeomorphism. Let  $y_k = (1 + x_k)/2$ . Then  $T_\alpha^{k+1}: (y_{k+1}, y_k) \rightarrow (1/2, 1)$  is a diffeomorphism. It is clear that  $\tau_\alpha = k + 1$  on  $(y_{k+1}, y_k)$ , so the map  $F_\alpha$  has full branches on the intervals  $(y_{k+1}, y_k)$  for  $k \geq 0$ .

We index branches by  $r \in \mathcal{R} = \mathbb{N} \cup \{0\}$ , the  $r$ -th branch being the one on  $(y_{r+1}, y_r)$ . Let  $F_{\alpha,r}: [y_{r+1}, y_r] \rightarrow [0, 1]$  be the continuous extension of  $F_\alpha: (y_{r+1}, y_r) \rightarrow (0, 1)$ .

For notational convenience we introduce a function

$$\text{logg}(r) = \begin{cases} 1 & r \leq e \\ \log(r) & r > e \end{cases}.$$

Let  $\varphi \in C^1[0, 1]$  be an observable; let

$$\Phi_\alpha = \sum_{k=0}^{\tau_\alpha-1} \varphi \circ T_\alpha^k \quad (3.1)$$

be the corresponding observable for the induced system.

**Theorem 3.1.** *The family of maps  $F_\alpha = T_\alpha^{\tau_\alpha}: [1/2, 1] \rightarrow [1/2, 1]$  with observables  $\Phi_\alpha$  fits into the setup of Theorem 2.1 and Corollary 2.4 with branches indexed as above,  $\delta_r = K(r+1) \|\varphi\|_{C^1}$ ,  $\sigma = 1/2$ , and  $\gamma_r = K(\text{logg } r)^3$ , where  $K$  is a constant, depending only on  $\alpha_-$  and  $\alpha_+$ .*

The proof consists of verification of the assumptions of Theorem 2.1 and Corollary 2.4, and is carried out in Subsection 5.2. Here we use Theorem 3.1 to prove our main result — Theorem 1.1.

*Proof of Theorem 1.1.* The invariant measure  $\nu_\alpha$  for  $T_\alpha$  is related to the invariant measure  $\mu_\alpha$  for  $F_\alpha$  by Kac's formula:

$$\int \varphi d\nu_\alpha = \int \Phi_\alpha d\mu_\alpha / \int \tau_\alpha d\mu_\alpha,$$

where  $\Phi_\alpha$  is given by (3.1). Note that if  $\varphi \equiv 1$ , then  $\Phi_\alpha = \tau_\alpha$ . By Theorem 3.1, both integrals are continuously differentiable in  $\alpha$ . Also,  $\tau_\alpha \geq 1$ , so  $\int \tau_\alpha d\mu_\alpha \geq 1$ . Hence  $\int \varphi d\nu_\alpha$  is continuously differentiable in  $\alpha$ .  $\square$

## 4 Proof of Theorem 2.1

For  $h \in C^1(I)$  and  $\alpha \in [\alpha_-, \alpha_+]$  define  $Q_\alpha h = \partial_\alpha(P_\alpha h)$ , if the derivative exists. Denote

$$\mathcal{Q}_h(\alpha) = Q_\alpha h, \quad \mathcal{P}_h(\alpha) = P_\alpha h.$$

### 4.1 Outline of the proof

The proof consists of three steps:

**(a) Continuity (Subsection 4.2).** We show that for  $i = 1, 2$  the linear operators

$$P_\alpha: C^i(I) \rightarrow C^i(I) \quad \text{and} \quad Q_\alpha: C^i(I) \rightarrow C^{i-1}(I)$$

are well defined, and their norms are bounded uniformly in  $\alpha$ . Plus, they continuously depend on  $\alpha$  in the following sense: for each  $h \in C^i(I)$  the maps  $\mathcal{P}_h: [\alpha_-, \alpha_+] \rightarrow C^i(I)$  and  $\mathcal{Q}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$  are continuous. Moreover, the map  $\mathcal{P}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$  is continuously differentiable, and its derivative is  $\mathcal{Q}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$ .

In addition,  $\int Q_\alpha h \, dm = 0$  for every  $h \in C^1(I)$ .

**(b) Distortion bounds and coupling (Subsection 4.3).**

- If  $h$  is in  $C^i(I)$  for  $i = 1$  or  $2$ , and  $\int h \, dm = 0$ , then  $\|P_\alpha^k h\|_{C^i} \rightarrow 0$  exponentially fast, uniformly in  $\alpha$ .
- If  $h \in C^2(I)$  and  $\int h \, dm = 1$ , then  $h_\alpha = \lim_{k \rightarrow \infty} P_\alpha^k h = h + \sum_{k=0}^{\infty} P_\alpha^k (P_\alpha h - h)$
- The series above converges exponentially fast in  $C^2$ , and  $\|h_\alpha\|_{C^2}$  is bounded on  $[\alpha_-, \alpha_+]$ .
- The map  $\alpha \mapsto h_\alpha$  from  $[\alpha_-, \alpha_+]$  to  $C^2(I)$  is continuous.

**(c) Computation of  $\partial_\alpha h_\alpha$ .** Fix  $\alpha \in [\alpha_-, \alpha_+]$ . Start with a formula, which holds for every  $\alpha, \beta$  and  $n$ :

$$P_\beta^n h_\alpha - P_\alpha^n h_\alpha = \sum_{k=0}^{n-1} P_\beta^k (P_\beta - P_\alpha) h_\alpha.$$

Since the terms in the above sum converge exponentially fast in  $C^2$ , we can take the limit  $n \rightarrow \infty$ . Note that  $P_\alpha^n h_\alpha = h_\alpha$  and  $\lim_{n \rightarrow \infty} P_\beta^n h_\alpha = h_\beta$ . Hence

$$h_\beta - h_\alpha = \sum_{k=0}^{\infty} P_\beta^k (P_\beta - P_\alpha) h_\alpha.$$

For fixed  $\alpha$ , recall that the map  $\beta \mapsto P_\beta h_\alpha$  from  $[\alpha_-, \alpha_+]$  to  $C^1(I)$  is continuously differentiable, and its derivative is the map  $\beta \mapsto Q_\beta h_\alpha$ , hence

$$(P_\beta - P_\alpha) h_\alpha = (\beta - \alpha) Q_\alpha h_\alpha + R_\beta,$$

with  $\|R_\beta\|_{C^1} = o(\beta - \alpha)$ . Note that both  $Q_\alpha h_\alpha$  and  $R_\beta$  have zero mean. Next,

$$h_\beta - h_\alpha = (\beta - \alpha) \sum_{k=0}^{\infty} P_\beta^k Q_\alpha h_\alpha + \sum_{k=0}^{\infty} P_\beta^k R_\beta.$$

Both series converge exponentially fast in  $C^1(I)$ , uniformly in  $\alpha$  and  $\beta$ , and the  $C^1$  norm of the second one is  $o(\alpha - \beta)$ .

Observe that the maps

$$\begin{array}{ccc} [\alpha_-, \alpha_+] \times C^1(I) & \longrightarrow & C^1(I) \\ \alpha, h & \longmapsto & P_\alpha h \end{array} \quad \text{and} \quad \begin{array}{ccc} [\alpha_-, \alpha_+] \times C^2(I) & \longrightarrow & C^1(I) \\ \alpha, h & \longmapsto & Q_\alpha h \end{array}$$

are continuous in  $\alpha$ , and continuous in  $h$  uniformly in  $\alpha$ , because  $P_\alpha$  and  $Q_\alpha$  are linear operators, bounded uniformly in  $\alpha$ . Thus both maps are jointly continuous in  $\alpha$  and  $h$ . Recall that the map  $\alpha \rightarrow h_\alpha$  from  $[\alpha_-, \alpha_+]$  to  $C^2(I)$  is continuous. Thus the map  $\alpha, \beta \mapsto P_\beta^k Q_\alpha h_\alpha$  from  $[\alpha_-, \alpha_+]^2$  to  $C^1(I)$  is continuous.

Therefore, in  $C^1(I)$  topology,

$$\lim_{\beta \rightarrow \alpha} \frac{h_\beta - h_\alpha}{\beta - \alpha} = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha,$$

and  $\sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$  continuously depends on  $\alpha$ . Note that the above also implies that  $\partial_\alpha h_\alpha = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$  (if understood pointwise).

Therefore the map  $\alpha \mapsto h_\alpha$  from  $[\alpha_-, \alpha_+]$  to  $C^1(I)$  is continuously differentiable, and its derivative is  $\alpha \mapsto \partial_\alpha h_\alpha = \sum_{k=0}^{\infty} P_\alpha^k Q_\alpha h_\alpha$ .

In the remainder of this section we make the above precise.

## 4.2 Continuity

**Lemma 4.1.** *Let  $K_6 = 4K_0(1 + K_0)$ . For each  $i = 1, 2$  and  $h \in C^i(I)$ :*

- a) *The map  $\mathcal{P}_h: [\alpha_-, \alpha_+] \rightarrow C^i(I)$  is continuous. Also,  $\|P_\alpha h\|_{C^i} \leq K_6 \|h\|_{C^i}$ .*
- b) *The map  $\mathcal{Q}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$  is continuous. Also,  $\|Q_\alpha h\|_{C^{i-1}} \leq K_6 \|h\|_{C^i}$ .*
- c) *The map  $\mathcal{Q}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$  is the derivative of the map  $\mathcal{P}_h: [\alpha_-, \alpha_+] \rightarrow C^{i-1}(I)$ .*
- d)  $\int Q_\alpha h \, dm = 0$ .

*Proof.* We do the case  $i = 2$ ; the case  $i = 1$  is similar and simpler.

- a) Let  $p_{\alpha,r} = (h \circ F_{\alpha,r}^{-1})G_{\alpha,r}$ . Then

$$\begin{aligned} p'_{\alpha,r} &= \pm(h' \circ F_{\alpha,r}^{-1})G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})G'_{\alpha,r}, \quad \text{and} \\ p''_{\alpha,r} &= (h'' \circ F_{\alpha,r}^{-1})G_{\alpha,r}^3 \pm 3(h' \circ F_{\alpha,r}^{-1})G_{\alpha,r}G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1})G''_{\alpha,r}, \end{aligned}$$

where the sign of  $\pm$  depends only on  $r$ . By assumptions A1, A2 and A3,  $\|p_{\alpha,r}\|_{C^2} \leq (4K_0 + 1)\|h\|_{C^2}\|G_{\alpha,r}\|_\infty$ .

Since  $p_{\alpha,r}$ ,  $p'_{\alpha,r}$  and  $p''_{\alpha,r}$  are jointly continuous in  $\alpha$  and  $\xi$ , we obtain that the map  $\alpha \mapsto p_{\alpha,r}$  from  $[\alpha_-, \alpha_+]$  to  $C^2(I)$  is continuous.

By assumption A7,  $\sum_r \|G_{\alpha,r}\|_\infty \leq K_0$ , so the map  $\alpha \mapsto P_\alpha h = \sum_r p_{\alpha,r}$  is continuous from  $[\alpha_-, \alpha_+]$  to  $C^2$ , and  $\|P_\alpha h\|_{C^2} \leq K_0(4K_0 + 1)\|h\|_{C^2}$ .

- b) Let  $q_{\alpha,r} = \partial_\alpha[(h \circ F_{\alpha,r}^{-1})G_{\alpha,r}]$ . We use the fact that  $F_{\alpha,r}^{-1}$ , as a function of  $\alpha$  and  $\xi$ , has continuous partial derivatives up to second order, to compute

$$q'_{\alpha,r} = \partial_\alpha[(h \circ F_{\alpha,r}^{-1})G_{\alpha,r}]' = \partial_\alpha[\pm(h' \circ F_{\alpha,r}^{-1})G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1})G'_{\alpha,r}].$$

We use assumptions A1, A2, A4, A5 and A6 to estimate

$$\|q_{\alpha,r}\|_{C^1} \leq \|G_{\alpha,r}\|_\infty(4 + K_0)\gamma_r\|h\|_{C^2}.$$



Since  $q_{\alpha,r}$  and  $q'_{\alpha,r}$  are jointly continuous in  $\alpha$  and  $\xi$ , we obtain that the map  $\alpha \mapsto q_{\alpha,r}$  from  $[\alpha_-, \alpha_+]$  to  $C^1(I)$  is continuous.

By assumption A7,  $\sum_r \gamma_r \|G_{\alpha,r}\|_\infty \leq K_0$ , so the map  $\alpha \mapsto Q_\alpha h = \sum_r q_{\alpha,r}$  is continuous from  $[\alpha_-, \alpha_+]$  to  $C^1$ , and  $\|Q_\alpha h\|_{C^1} \leq K_0(4 + K_0)\|h\|_{C^2}$ .

- c) Note that  $(Q_\alpha h)(\xi)$  and  $(Q_\alpha h)'(\xi)$  are jointly continuous in  $\alpha$  and  $\xi$ . By definition of  $Q_\alpha$ , for every  $\xi$  and  $j = 0, 1$  we can write

$$\begin{aligned} (P_\beta h)^{(j)}(\xi) - (P_\alpha h)^{(j)}(\xi) &= \int_\alpha^\beta (Q_t h)^{(j)}(\xi) dt \\ &= (\beta - \alpha)(Q_\alpha h)^{(j)}(\xi) + \int_\alpha^\beta [(Q_t h)^{(j)}(\xi) - (Q_\alpha h)^{(j)}(\xi)] dt. \end{aligned}$$

Fix  $\alpha$ . Since  $\lim_{t \rightarrow \alpha} \|Q_t h - Q_\alpha h\|_{C^1} = 0$ , the integral on the right is  $o(\beta - \alpha)$  uniformly in  $\xi$ . Therefore,

$$\|P_\beta h - P_\alpha h - (\beta - \alpha)Q_\alpha h\|_{C^1} = o(\beta - \alpha),$$

thus  $\mathcal{Q}_h: [\alpha_-, \alpha_+] \rightarrow C^1(I)$  is the derivative of  $\mathcal{P}_h: [\alpha_-, \alpha_+] \rightarrow C^1(I)$ .

- d) To prove that  $\int Q_\alpha h dm = 0$ , we differentiate the identity  $\int P_\alpha h dm = \int h dm$  by  $\alpha$ :

$$\int Q_\alpha h dm = \frac{d}{d\alpha} \int P_\alpha h dm = 0,$$

the order of differentiation and integration can be changed because both  $P_\alpha h$  and  $\partial_\alpha(P_\alpha h) = Q_\alpha h$  are jointly continuous in  $\alpha$  and  $\xi$ .

□

### 4.3 Distortion bounds and coupling

If  $h \in C^1$  and  $h$  is positive, denote  $\|h\|_L = \|h'/h\|_\infty$ . If also  $h \in C^2$ , denote  $\|h\|_P = \|h''/h\|_\infty$ .

**Lemma 4.2** (Distortion bounds). *If  $h \in C^1$  and  $h > 0$ , then*

$$\|P_\alpha h\|_L \leq \sigma \|h\|_L + K_0. \quad (4.1)$$

*If also  $h \in C^2$ , then*

$$\|P_\alpha h\|_P \leq \sigma^2 \|h\|_P + 3\sigma K_0 \|h\|_L + K_0. \quad (4.2)$$

*Proof.* Recall that  $P_\alpha h = \sum_r (h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}$  and  $(F_{\alpha,r}^{-1})' = \pm G_{\alpha,r}$ , where the sign depends only on  $r$ . Inequality (4.1) follows from the following computation:

$$\begin{aligned} \left| \frac{(P_\alpha h)'}{P_\alpha h} \right| &= \left| \frac{\sum_r \pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1}) G'_{\alpha,r}}{\sum_r (h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}} \right| \\ &\leq \max_r \left| \frac{\pm (h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^2 + (h \circ F_{\alpha,r}^{-1}) G'_{\alpha,r}}{(h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}} \right| \leq \max_r \left( \frac{|h' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} G_{\alpha,r} + \frac{|G'_{\alpha,r}|}{G_{\alpha,r}} \right) \\ &\leq \max_r (\|h\|_L \|G_{\alpha,r}\|_\infty + \|G_{\alpha,r}\|_L) \end{aligned}$$

and assumptions A1 and A2.

Next,

$$(P_\alpha h)'' = \sum_r [(h'' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^3 \pm 3(h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r} G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1}) G''_{\alpha,r}].$$

Thus

$$\begin{aligned} \left| \frac{(P_\alpha h)''}{P_\alpha h} \right| &\leq \max_r \frac{|(h'' \circ F_{\alpha,r}^{-1}) G_{\alpha,r}^3 \pm 3(h' \circ F_{\alpha,r}^{-1}) G_{\alpha,r} G'_{\alpha,r} + (h \circ F_{\alpha,r}^{-1}) G''_{\alpha,r}|}{(h \circ F_{\alpha,r}^{-1}) G_{\alpha,r}} \\ &\leq \max_r \left( \frac{|h'' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} G_{\alpha,r}^2 + 3 \frac{|h' \circ F_{\alpha,r}^{-1}|}{h \circ F_{\alpha,r}^{-1}} \frac{|G'_{\alpha,r}|}{G_{\alpha,r}} G_{\alpha,r} + \frac{|G''_{\alpha,r}|}{G_{\alpha,r}} \right) \\ &\leq \max_r (\|h\|_P \|G_{\alpha,r}\|_\infty^2 + 3\|h\|_L \|G_{\alpha,r}\|_L \|G_{\alpha,r}\|_\infty + \|G_{\alpha,r}\|_P). \end{aligned}$$

The inequality (4.2) follows from the above and assumptions A1, A2 and A3.  $\square$

Let  $K_L > 0$ ,  $K_P > 0$  and  $\theta \in (0, 1)$  be constants satisfying

$$\begin{aligned} K_L(1 - \theta e^{|I|K_L}) &> \sigma K_L + K_0, \quad \text{and} \\ K_P(1 - \theta e^{|I|K_L}) &> \sigma^2 K_P + 3\sigma K_0 K_L + K_0, \end{aligned} \tag{4.3}$$

where  $|I|$  means the length of the interval  $I$ . It is clear that such constants can be chosen, because  $\sigma < 1$ .

**Definition 4.3.** We say that a function  $h$  is *regular*, if it is positive, belongs in  $C^1(I)$ , and  $\|h\|_L \leq K_L$ . If in addition  $h \in C^2(I)$  and  $\|h\|_P \leq K_P$ , we say that  $h$  is *superregular*.

*Remark 4.4.* It readily follows from Lemma 4.2 that if  $h$  is a regular function, then so is  $P_\alpha h$ . If  $h$  is superregular, then so is  $P_\alpha h$ .

*Remark 4.5.* We observe that regular functions are explicitly bounded from above and from below. If  $h$  is regular, so  $\|h'/h\|_\infty < K_L$ , then  $h(x_1)/h(x_2) \leq e^{|I|K_L}$  for all  $x_1, x_2$ . Also, there is  $\hat{x} \in I$  such that  $h(\hat{x}) = \int h \, dm$ , hence

$$e^{-|I|K_L} \int h \, dm \leq h(x) \leq e^{|I|K_L} \int h \, dm \quad \text{for all } x \in I. \tag{4.4}$$

**Lemma 4.6.** Assume that  $h$  is regular. Let  $g = P_\alpha h - \theta \int h \, dm$ . Then  $g$  is regular. If  $h$  is superregular, then so is  $g$ .

*Proof.* Since  $P_\alpha h \geq e^{-|I|K_L} \int P_\alpha h \, dm = e^{-|I|K_L} \int h \, dm$  by equation (4.4),

$$g = P_\alpha h \left( 1 - \frac{\theta \int h \, dm}{P_\alpha h} \right) \geq P_\alpha h (1 - \theta e^{|I|K_L}).$$

Thus  $g > 0$ . By Lemma 4.2 and equation (4.3)

$$\|g\|_L = \left\| \frac{g'}{g} \right\|_\infty = \left\| \frac{(P_\alpha h)'}{g} \right\|_\infty \leq \left\| \frac{(P_\alpha h)'}{P_\alpha h} \right\|_\infty \frac{1}{1 - \theta e^{|I|K_L}} = \|P_\alpha h\|_L \frac{1}{1 - \theta e^{|I|K_L}} \leq K_L.$$

Hence  $g$  is regular. An analogous proof works for  $\|g\|_P$ .  $\square$

**Lemma 4.7** (Coupling Lemma). *Let  $f$  and  $g$  be two regular functions with  $\int f dm = \int g dm = M$ . Let  $f_0 = f$  and  $g_0 = g$ , and define*

$$f_{n+1} = P_\alpha f_n - \theta \int f_n dm, \quad g_{n+1} = P_\alpha g_n - \theta \int g_n dm.$$

*Then for all  $n$*

$$P_\alpha^n(f - g) = f_n - g_n,$$

*where  $f_n$  and  $g_n$  are regular, and  $\int f_n dm = \int g_n dm = (1 - \theta)^n M$ .*

*In particular,  $\|f_n\|_\infty, \|g_n\|_\infty \leq (1 - \theta)^n e^{|I|K_L} M$ , and*

$$\|f'_n\|_\infty, \|g'_n\|_\infty \leq K_L(1 - \theta)^n e^{|I|K_L} M.$$

*If in addition  $f$  and  $g$  are superregular, then*

$$\|f''_n\|_\infty, \|g''_n\|_\infty \leq K_P(1 - \theta)^n e^{|I|K_L} M.$$

*Proof.* The proof of  $\int f_n dm = \int g_n dm = (1 - \theta)^n M$  is by induction.

By equation (4.4),  $\|f\|_\infty$  and  $\|g\|_\infty$  are bounded by  $(1 - \theta)^n e^{|I|K_L} M$ . Note that if  $h$  is a regular function, then  $\|h'\|_\infty \leq K_L \|h\|_\infty$ , and if it is superregular, then also  $\|h''\|_\infty \leq K_P \|h\|_\infty$ . The bounds on  $\|f'\|_\infty, \|g'\|_\infty, \|f''\|_\infty, \|g''\|_\infty$  follow.  $\square$

**Corollary 4.8.** *There is a constant  $K_5$  such that if  $h \in C^i(I)$  for  $i = 1$  or  $2$ , and  $h$  has mean zero, then*

$$\|P_\alpha^n h\|_{C^i} \leq K_5(1 - \theta)^n \|h\|_{C^i}.$$

*Proof.* We can represent  $h = (h + c) - c$ , where  $c = \|h\|_{C^i}(1 + \max(K_L^{-1}, K_P^{-1}))$ . Then

$$\left\| \frac{h'}{h + c} \right\|_\infty \leq \frac{\|h\|_{C^i}}{-\|h\|_{C^i} + c} = \frac{1}{\max(K_L^{-1}, K_P^{-1})} = \min(K_L, K_P),$$

and so  $h + c$  is regular. If  $i = 2$ , then the same identity with  $h''$  in place of  $h'$  also holds true, so also  $h + c$  is superregular.

By Lemma 4.7 applied to  $f = h + c$  and  $g = c$ ,

$$P_\alpha^n h = f_n - g_n,$$

where

$$\begin{aligned} \|f_n\|_{C^i}, \|g_n\|_{C^i} &\leq \max(1, K_L, K_P)(1 - \theta)^n e^{|I|K_L} c \\ &= (1 - \theta)^n [\max(1, K_L, K_P) e^{|I|K_L} (1 + \max(K_L^{-1}, K_P^{-1}))] \|h\|_{C^i} \\ &= (1 - \theta)^n \frac{K_5}{2} \|h\|_{C^i}. \end{aligned}$$

Thus  $\|P_\alpha^n h\|_{C^i} \leq (1 - \theta)^n K_5 \|h\|_{C^i}$ , where

$$K_5 = 2 \max(1, K_L, K_P)(1 + \max(K_L^{-1}, K_P^{-1})) e^{|I|K_L}.$$

$\square$

**Corollary 4.9.** *For any  $h \in C^2(I)$  with  $\int h \, dm = 1$*

$$h_\alpha = \lim_{n \rightarrow \infty} P_\alpha^n h = h + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha h - h).$$

*The series converges exponentially fast in  $C^2$ . The  $C^2$  norm of  $h_\alpha$  is bounded by  $K_1 = 1 + 2\theta^{-1}e^{|I|K_L} \max(1, K_L, K_P)$ .*

*Proof.* Let

$$f = 1 + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha 1 - 1).$$

Since 1 is a superregular function, so is  $P_\alpha 1$ , and by Lemma 4.7 applied to  $f = P_\alpha 1$  and  $g = 1$ , we have that  $\|P_\alpha^n (P_\alpha 1 - 1)\|_{C^2} \leq 2(1 - \theta)^n e^{|I|K_L} \max(1, K_L, K_P)$ . Thus the series above converges exponentially fast in  $C^2(I)$  and  $\|h\|_{C^2} \leq K_1$ .

Since  $1 + \sum_{n=0}^N P_\alpha^n (P_\alpha 1 - 1) = P_\alpha^{N+1} 1$ , we have  $f = \lim_{n \rightarrow \infty} P_\alpha^n 1$ . Thus  $f$  is invariant under  $P_\alpha$ . It is clear that  $\int f \, dm = 1$ . Thus  $h_\alpha = f$ .

By Corollary 4.8, the  $C^2$  norm of  $P_\alpha^n (h - h_\alpha) = (P_\alpha^n h) - h_\alpha$  decreases exponentially with  $n$ , thus  $h_\alpha = \lim_{n \rightarrow \infty} P_\alpha^n h = h + \sum_{n=0}^{\infty} P_\alpha^n (P_\alpha h - h)$ .  $\square$

**Corollary 4.10.** *The map  $\alpha \mapsto h_\alpha$  from  $[\alpha_-, \alpha_+]$  to  $C^2(I)$  is continuous.*

*Proof.* Using Corollary 4.9, write for  $N \in \mathbb{N}$ :

$$h_\alpha = 1 + \sum_{n=0}^{N-1} P_\alpha^n (P_\alpha 1 - 1) + \sum_{n=N}^{\infty} P_\alpha^n (P_\alpha 1 - 1).$$

The  $C^2$  norm of the second sum is exponentially small in  $N$ , uniformly in  $\alpha$ . By Lemma 4.1, a map  $\alpha \mapsto P_\alpha^n (P_\alpha h - h)$  from  $[\alpha_-, \alpha_+]$  to  $C^2(I)$  is continuous for every  $n$ . Thus the first sum depends on  $\alpha$  continuously. Since the choice of  $N$  is arbitrary, the result follows.  $\square$

## 5 Proofs of Theorems 3.1 and 1.2

In this section we prove technical statements about the family of maps  $T_\alpha$ , defined by equation (1.1). We use notations introduced in Section 3.

In Subsection 5.1 we introduce necessary notations and prove a number of technical lemmas, in Subsections 5.2 and 5.3 we use the accumulated knowledge to prove Theorems 3.1 and 1.2.

### 5.1 Technical Lemmas

We use notation  $\mathbf{C}$  for various nonnegative constants, which only depend on  $\alpha_-$  and  $\alpha_+$ , and may change from line to line, and within one expression if used twice. Recall the definition of  $y_r$  from the beginning of Section 3.

It is clear that  $T_\alpha(x)$ , as a function of  $\alpha$  and  $x$ , has continuous partial derivatives of all orders in  $\alpha, x \in [\alpha_-, \alpha_+] \times (0, 1/2]$ , and so do  $F_{\alpha,r}(x)$  and  $F_{\alpha,r}^{-1}(x)$  on  $[\alpha_-, \alpha_+] \times [y_{r+1}, y_r]$  and  $[\alpha_-, \alpha_+] \times [1/2, 1]$  respectively.

Let  $E_\alpha: [0, 1/2] \rightarrow [0, 1]$ ,  $E_\alpha x = T_\alpha x$  be the left branch of the map  $T_\alpha$ . Note that  $E_\alpha$  is invertible. Let  $z \in [0, 1]$  and write, for notational convenience,  $z_r = E_\alpha^{-r}(z)$ . Then  $F_{\alpha,r}(z) = E_\alpha^r(T_\alpha(z)) = E_\alpha^r(2z - 1)$  for  $z \in [y_{r+1}, y_r]$ , and for  $z \in [1/2, 1]$

$$T_\alpha(F_{\alpha,r}^{-1}(z)) = 2F_{\alpha,r}^{-1}(z) - 1 = z_r. \quad (5.1)$$

By  $(\cdot)'$  we denote the derivative with respect to  $z$ . Let  $G_{\alpha,r}$  be defined as in Theorem 2.1. Then for  $z \in [1/2, 1]$

$$G_{\alpha,r}(z) = (F_{\alpha,r}^{-1})'(z) = z'_r/2. \quad (5.2)$$

We do all the analysis in terms of  $z_r$ , and the relation to  $G_{\alpha,r}$  and  $F_{\alpha,r}^{-1}$  is given by equations (5.1) and (5.2).

*Remark 5.1.* By construction,  $z_0 = z$ ,  $z'_0 = 1$  and  $z''_0 = 0$ . Note that  $z_r \leq 1/2$  for  $r \geq 1$ . Also

$$z_r = z_{r+1}(1 + 2^\alpha z_{r+1}^\alpha), \quad (5.3)$$

$$z'_r = [1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha] z'_{r+1}, \quad (5.4)$$

$$z'_r = \prod_{j=1}^r [1 + (\alpha + 1)2^\alpha z_j^\alpha]^{-1}. \quad (5.5)$$

Our analysis is built around the following estimate:

**Lemma 5.2.** *For  $r \geq 1$*

$$\frac{1}{z_0^{-\alpha} + r\alpha 2^\alpha} \leq z_r^\alpha \leq \frac{1}{z_0^{-\alpha} + r\alpha(1 - \alpha)2^{\alpha-1}}.$$

*In particular,*

$$\frac{\mathbf{C} z_0^\alpha}{r} \leq z_r^\alpha \leq \frac{\mathbf{C}}{r} \quad \text{and} \quad -\log z_r \leq \mathbf{C} [\log r - \log z_0].$$

*Proof.* Transform equation (5.3) into

$$z_{r+1}^{-\alpha} = z_r^{-\alpha} + \frac{1 - (1 + 2^\alpha z_{r+1}^\alpha)^{-\alpha}}{z_{r+1}^\alpha}.$$

Then

$$z_r^{-\alpha} = z_0^{-\alpha} + \sum_{j=1}^r \frac{1 - (1 + 2^\alpha z_j^\alpha)^{-\alpha}}{z_j^\alpha}. \quad (5.6)$$

For all  $t \in (0, 1)$  and all  $\alpha \in (0, 1)$

$$1 - \alpha t \leq (1 + t)^{-\alpha} \leq 1 - \alpha t + \frac{\alpha(\alpha + 1)}{2} t^2.$$

Since  $z_j \in (0, 1/2]$  for  $j \geq 1$ , using the above inequality with  $t = 2^\alpha z_j^\alpha$ , we obtain

$$\alpha(1 - \alpha)2^{\alpha-1} \leq \frac{1 - (1 + 2^\alpha z_j^\alpha)^{-\alpha}}{z_j^\alpha} \leq \alpha 2^\alpha.$$

By equation (5.6),

$$r \alpha (1 - \alpha) 2^{\alpha-1} \leq z_r^{-\alpha} - z_0^{-\alpha} \leq r \alpha 2^\alpha.$$

for  $r \geq 1$ . Write

$$\frac{z_0^\alpha}{r} \frac{1}{1 + \alpha 2^\alpha} \leq \frac{z_0^\alpha}{r} \frac{1}{r^{-1} + z_0^\alpha \alpha 2^\alpha} = \frac{1}{z_0^{-\alpha} + r \alpha 2^\alpha} \leq z_r^\alpha \leq \frac{1}{z_0^{-\alpha} + r \alpha (1 - \alpha) 2^{\alpha-1}}.$$

The result follows.  $\square$

**Lemma 5.3.**  $z'_0 = 1$  and

$$0 \leq z'_r \leq \mathbf{C} (1 + r z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \leq \mathbf{C} r^{-(\alpha+1)/\alpha} z_0^{-(\alpha+1)}$$

for  $r \geq 1$ .

*Proof.* By Remark 5.1,  $z'_0 = 1$ . Let  $r \geq 1$ . Using the inequality

$$\frac{1}{1+t} \leq \exp(-t + t^2) \quad \text{for } t \geq 0$$

on equation (5.5) we obtain

$$0 \leq z'_r = \prod_{j=1}^r \frac{1}{1 + (\alpha + 1) 2^\alpha z_j^\alpha} \leq \exp \left( - \sum_{j=1}^r (\alpha + 1) 2^\alpha z_j^\alpha + \sum_{j=1}^r ((\alpha + 1) 2^\alpha z_j^\alpha)^2 \right). \quad (5.7)$$

By Lemma 5.2,  $(z_j^\alpha)^2 \leq \mathbf{C}/j^2$ , thus the second sum under the exponent is bounded by  $\mathbf{C}$ . Also by Lemma 5.2,

$$\begin{aligned} \sum_{j=1}^r z_j^\alpha &\geq \sum_{j=1}^r \frac{1}{z_0^{-\alpha} + j \alpha 2^\alpha} \geq \int_1^r \frac{z_0^\alpha}{1 + t z_0^\alpha \alpha 2^\alpha} dt - \mathbf{C} \\ &= \frac{1}{\alpha 2^\alpha} \log(1 + t z_0^\alpha \alpha 2^\alpha) \Big|_{t=1}^{t=r} - \mathbf{C} \\ &\geq \frac{\log(1 + r z_0^\alpha \alpha 2^\alpha)}{\alpha 2^\alpha} - \mathbf{C} \end{aligned}$$

Thus

$$-(\alpha + 1) 2^\alpha \sum_{j=1}^r z_j^\alpha \leq -\frac{\alpha + 1}{\alpha} \log(1 + r z_0^\alpha \alpha 2^\alpha) + \mathbf{C},$$

and by equation (5.7),

$$z'_r \leq \mathbf{C} (1 + r z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \leq \mathbf{C} (r z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \leq \mathbf{C} r^{-(\alpha+1)/\alpha} z_0^{-(\alpha+1)}.$$

$\square$

**Lemma 5.4.**  $0 \leq -z''_r/z'_r \leq \mathbf{C} z_0^{-2}/\max(r, 1)$ .

*Proof.* Differentiating both sides of the equation (5.4), we obtain

$$z_r'' = \alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} (z_{r+1}')^2 + (1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha) z_{r+1}''. \quad (5.8)$$

Dividing the above by  $z_r' = [1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha] z_{r+1}'$  we get

$$\frac{z_r''}{z_r'} = \frac{\alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}'}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha} + \frac{z_{r+1}''}{z_{r+1}'}$$

Recall that  $z_0''/z_0' = 0$  and  $z_r' \geq 0$ , thus  $z_r'' \leq 0$  for all  $r$ . By Lemmas 5.2 and 5.3 we have

$$0 \leq z_r^{\alpha-1} z_r' \leq \mathbf{C} \left( \frac{z_0^\alpha}{r} \right)^{(\alpha-1)/\alpha} r^{-\frac{\alpha+1}{\alpha}} z_0^{-(\alpha+1)} \leq \mathbf{C} r^{-2} z_0^{-2}$$

for  $r \geq 1$ . Thus

$$0 \leq \frac{z_r''}{z_r'} - \frac{z_{r+1}''}{z_{r+1}'} \leq \mathbf{C} (r + 1)^{-2} z_0^{-2}.$$

The result follows.  $\square$

**Lemma 5.5.**  $|z_r'''/z_r'| \leq \mathbf{C} z_0^{-\alpha-4}/\max(r^2, 1)$ .

*Proof.* Differentiate the equation (5.8). This results in

$$\begin{aligned} z_r''' &= (\alpha - 1)\alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-2} (z_{r+1}')^3 + 3\alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}' z_{r+1}'' \\ &\quad + (1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha) z_{r+1}'''. \end{aligned}$$

Dividing the above by  $z_r' = [1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha] z_{r+1}'$  we get

$$\frac{z_r'''}{z_r'} = \frac{(\alpha - 1)\alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-2} (z_{r+1}')^2}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha} + \frac{3\alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} z_{r+1}'}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha} \frac{z_{r+1}''}{z_{r+1}'} + \frac{z_{r+1}'''}{z_{r+1}'}$$

Using Lemmas 5.2, 5.3 and 5.4 we bound the first two terms in the right hand side above by  $\mathbf{C} (r + 1)^{-3} z_0^{-\alpha-4}$  and  $\mathbf{C} (r + 1)^{-3} z_0^{-4}$  respectively. Thus

$$\left| \frac{z_r'''}{z_r'} - \frac{z_{r+1}'''}{z_{r+1}'} \right| \leq \mathbf{C} (r + 1)^{-3} z_0^{-\alpha-4}.$$

Since  $z_0'''/z_0' = 0$ , the result follows.  $\square$

**Lemma 5.6.**  $\partial_\alpha z_0 = 0$  and for  $r \geq 1$

$$\begin{aligned} 0 \leq \frac{\partial_\alpha z_r}{z_r} &\leq \mathbf{C} \log(r z_0^\alpha) [\log r - \log z_0] \quad \text{and} \\ 0 \leq \partial_\alpha z_r &\leq \mathbf{C} \frac{\log(r z_0^\alpha)}{r^{1/\alpha}} [\log r - \log z_0]. \end{aligned}$$

*Proof.* Since  $z_0 = z$  does not depend on  $\alpha$ ,  $\partial_\alpha z_0 = 0$ .

Differentiating the identity  $z_{r+1}(1 + 2^\alpha z_{r+1}^\alpha) = z_r$  by  $\alpha$  we obtain a recursive relation

$$\partial_\alpha z_{r+1} = \frac{\partial_\alpha z_r + 2^\alpha z_{r+1}^{\alpha+1} (-\log(2 z_{r+1}))}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha}.$$

Since  $z_{r+1} \leq 1/2$  for all  $r$ , it follows that  $\partial_\alpha z_r \geq 0$  for all  $r$ . It is convenient to rewrite the above, dividing by  $z_{r+1}$  and using  $z_{r+1}(1 + 2^\alpha z_{r+1}^\alpha) = z_r$ :

$$\frac{\partial_\alpha z_{r+1}}{z_{r+1}} = \frac{(1 + 2^\alpha z_{r+1}^\alpha) \frac{\partial_\alpha z_r}{z_r} + 2^\alpha z_{r+1}^\alpha (-\log(2z_{r+1}))}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha},$$

which implies

$$\frac{\partial_\alpha z_{r+1}}{z_{r+1}} \leq \frac{\partial_\alpha z_r}{z_r} + 2^\alpha z_{r+1}^\alpha (-\log(2z_{r+1})).$$

By Lemma 5.2,

$$2^\alpha z_r^\alpha (-\log(2z_r)) \leq \mathbf{C} \frac{\log r - \log z_0}{z_0^{-\alpha} + r\alpha(1 - \alpha)2^{\alpha-1}}.$$

Hence

$$\begin{aligned} \frac{\partial_\alpha z_r}{z_r} &\leq \sum_{j=1}^r 2^\alpha z_j^\alpha (-\log(2z_j)) \leq \mathbf{C} \int_1^r \frac{\log t - \log z_0}{z_0^{-\alpha} + t\alpha(1 - \alpha)2^{\alpha-1}} dt \\ &\leq \mathbf{C} \log(rz_0^\alpha) [\log(rz_0^\alpha) - \log z_0]. \end{aligned} \quad (5.9)$$

The first part of the lemma follows. To prove the second part, observe that by Lemma 5.2,  $z_r \leq \mathbf{C} r^{-1/\alpha}$ .  $\square$

**Lemma 5.7.**  $|(\partial_\alpha z'_r)/z'_r| \leq \mathbf{C} [\log(rz_0^\alpha)]^2 [\log r - \log z_0]$ .

*Proof.* Note that  $\partial_\alpha z'_0 = 0$ , because  $z_0 = z$  does not depend on  $\alpha$ .

Differentiate equation (5.4) by  $\alpha$ . This results in

$$\begin{aligned} \partial_\alpha z'_r &= (2^\alpha z_{r+1}^\alpha + (\alpha + 1)2^\alpha z_{r+1}^\alpha \log(2z_{r+1}) + \alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} \partial_\alpha z_{r+1}) z'_{r+1} \\ &\quad + (1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha) \partial_\alpha z'_{r+1} \end{aligned}$$

Dividing the above by  $z'_r = [1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha] z'_{r+1}$  we get

$$\frac{\partial_\alpha z'_r}{z'_r} = \frac{2^\alpha z_{r+1}^\alpha + (\alpha + 1)2^\alpha z_{r+1}^\alpha \log(2z_{r+1}) + \alpha(\alpha + 1)2^\alpha z_{r+1}^{\alpha-1} \partial_\alpha z_{r+1}}{1 + (\alpha + 1)2^\alpha z_{r+1}^\alpha} + \frac{\partial_\alpha z'_{r+1}}{z'_{r+1}}.$$

For  $r \geq 1$  Lemmas 5.2 and 5.6 give  $|z_r^\alpha| \leq \mathbf{C}/(z_0^{-\alpha} + r\alpha(1 - \alpha)2^{\alpha-1})$ ,

$$\begin{aligned} |z_r^\alpha \log z_r| &\leq \mathbf{C} \frac{\log r - \log z_0}{z_0^{-\alpha} + r\alpha(1 - \alpha)2^{\alpha-1}} \quad \text{and} \\ |z_r^{\alpha-1} \partial_\alpha z_r| &= \left| z_r^\alpha \frac{\partial_\alpha z_r}{z_r} \right| \leq \mathbf{C} \frac{\log(rz_0^\alpha) (\log r - \log z_0)}{z_0^{-\alpha} + r\alpha(1 - \alpha)2^{\alpha-1}}. \end{aligned}$$

Therefore

$$\left| \frac{\partial_\alpha z'_r}{z'_r} - \frac{\partial_\alpha z'_{r+1}}{z'_{r+1}} \right| \leq \mathbf{C} \frac{\log((r+1)z_0^\alpha) (\log(r+1) - \log z_0)}{z_0^{-\alpha} + (r+1)\alpha(1 - \alpha)2^{\alpha-1}}.$$

Thus

$$\left| \frac{\partial_\alpha z'_r}{z'_r} \right| \leq \mathbf{C} \int_1^r \frac{\log(tz_0^\alpha) (\log t - \log z_0)}{z_0^{-\alpha} + t\alpha(1 - \alpha)2^{\alpha-1}} dt \leq \mathbf{C} [\log(rz_0^\alpha)]^2 (\log r - \log z_0).$$

$\square$



**Lemma 5.8.**  $|(\partial_\alpha z_r'')/z_r'| \leq \mathbf{C} z_0^{-2} (1 - \log z_0)$ .

*Proof.* Differentiate both sides of the equation (5.8) by  $\alpha$ . This gives

$$\begin{aligned} \partial_\alpha z_r'' = & [2\alpha + 1 + \alpha(\alpha + 1) \log(2z_{r+1})] 2^\alpha z_{r+1}^{\alpha-1} (z_{r+1}')^2 \\ & + (\alpha - 1)\alpha(\alpha + 1) 2^\alpha z_{r+1}^{\alpha-2} (z_{r+1}')^2 \partial_\alpha z_{r+1} + 2\alpha(\alpha + 1) 2^\alpha z_{r+1}^{\alpha-1} z_{r+1}' \partial_\alpha z_{r+1}' \\ & + (1 + (\alpha + 1) \log(2z_{r+1})) 2^\alpha z_{r+1}^\alpha z_{r+1}'' + \alpha(\alpha + 1) 2^\alpha z_{r+1}^{\alpha-1} z_{r+1}'' \partial_\alpha z_{r+1} \\ & + (1 + (\alpha + 1) 2^\alpha z_{r+1}^\alpha) \partial_\alpha z_{r+1}''. \end{aligned}$$

Dividing the above by  $z_r' = [1 + (\alpha + 1) 2^\alpha z_{r+1}^\alpha] z_{r+1}'$  and using Lemma 5.2 to bound  $z_{r+1}$ , Lemma 5.3 to bound  $z_r'$ , Lemma 5.4 to bound  $z_r''/z_r'$ , Lemma 5.6 to bound  $\partial_\alpha z_r$  and Lemma 5.7 to bound  $\partial_\alpha z_r'/z_r'$ , we obtain for  $r \geq 0$ :

$$\begin{aligned} |z_r^{\alpha-1} z_r' \log z_r| & \leq \mathbf{C} r^{-2} z_0^{-2} (\log r - \log z_0), \\ |z_r^{\alpha-2} z_r' \partial_\alpha z_r| & \leq \mathbf{C} r^{-2} z_0^{-2} \log(r z_0^\alpha) (\log r - \log z_0), \\ |z_r^{\alpha-1} \partial_\alpha z_r'| & \leq \mathbf{C} r^{-2} z_0^{-2} [\log(r z_0^\alpha)]^2 (\log r - \log z_0), \\ |z_r^\alpha z_r'' \log z_r/z_r'| & \leq \mathbf{C} r^{-2} z_0^{-2} (\log r - \log z_0), \\ |z_r^{\alpha-1} z_r'' (\partial_\alpha z_r)/z_r'| & \leq \mathbf{C} r^{-2} z_0^{-2} \log(r z_0^\alpha) (\log r - \log z_0). \end{aligned}$$

Hence for  $r \geq 1$

$$\left| \frac{\partial_\alpha z_r''}{z_r'} - \frac{\partial_\alpha z_{r-1}''}{z_{r-1}'} \right| \leq \mathbf{C} r^{-2} z_0^{-2} [\log(r z_0^\alpha)]^2 (\log r - \log z_0).$$

Recall that  $\partial_\alpha z_0'' = 0$ . Then

$$\left| \frac{\partial_\alpha z_r''}{z_r'} \right| \leq z_0^{-2} \sum_{j=1}^r j^{-2} [\log(j z_0^\alpha)]^2 [\log j - \log z_0] \leq \mathbf{C} z_0^{-2} (1 - \log z_0).$$

□

## 5.2 Proof of Theorem 3.1

The verification of assumptions of Theorem 2.1 is as follows. Since  $G_{\alpha,r}$  and  $F_{\alpha,r}^{-1}$  are defined on  $[1/2, 1]$ , we use that  $z = z_0 \geq 1/2$  in the bounds below. Now,

A1. By equation (5.5),  $z_r' \leq 1$ , thus  $\|G_{\alpha,r}\|_\infty \leq 1/2$ .

A2. By Lemma 5.4,  $|z_r''/z_r'| \leq \mathbf{C}$ , thus  $\|G_{\alpha,r}'/G_{\alpha,r}\|_\infty \leq \mathbf{C}$ .

A3. By Lemma 5.5,  $|z_r'''/z_r'| \leq \mathbf{C}$ , thus  $\|G_{\alpha,r}''/G_{\alpha,r}\|_\infty \leq \mathbf{C}$ .

A4. By Lemma 5.6,  $|\partial_\alpha z_r| \leq \mathbf{C} r^{-1/\alpha} (\log r)^2$ , and by equation (5.1) we have

$$\|\partial_\alpha F_{\alpha,r}^{-1}\|_\infty \leq \mathbf{C} r^{-1/\alpha} (\log r)^2 \leq \mathbf{C} (\log r)^2.$$

A5. By Lemma 5.7,  $|(\partial_\alpha z_r')/z_r'| \leq \mathbf{C} (\log r)^3$ , thus  $\|(\partial_\alpha G_{\alpha,r})/G_{\alpha,r}\|_\infty \leq \mathbf{C} (\log r)^3$ .

A6. By Lemma 5.8,  $|(\partial_\alpha z_r'')/z_r'| \leq \mathbf{C}$ , thus  $\|(\partial_\alpha G_{\alpha,r}')/G_{\alpha,r}\|_\infty \leq \mathbf{C}$ .

A7. By Remark 5.1,  $z'_0 = 1$ , and by Lemma 5.3,  $|z'_r| \leq \mathbf{C} r^{-(\alpha+1)/\alpha}$  for  $r \geq 1$ , so

$$\sum_{r=0}^{\infty} \|G_{\alpha,r}\|_{\infty} (\log r)^3 = \frac{1}{2} \sum_{r=0}^{\infty} \sup_z |z'_r| \cdot (\log r)^3 \leq \frac{1}{2} + \mathbf{C} \sum_{r=1}^{\infty} \frac{(\log r)^3}{r^{1+1/\alpha}} \leq \mathbf{C}.$$

To verify the assumptions of the Corollary 2.4 — we have to show in addition that

- $\sum_{r=0}^{\infty} (r+1)(\log r)^3 \|G_{\alpha,r}\|_{\infty} \leq \mathbf{C}$ . By Lemma 5.3 and equation (5.2),

$$|G_{\alpha,r}(z)| = |z'_r|/2 \leq \mathbf{C} r^{-(\alpha+1)/\alpha},$$

thus

$$\sum_{r=0}^{\infty} (r+1)(\log r)^3 \|G_{\alpha,r}\|_{\infty} \leq \mathbf{C} \sum_{r=0}^{\infty} \frac{(\log r)^3}{r^{1/\alpha}} \frac{r+1}{r} \leq \mathbf{C}.$$

- $\|\partial_{\alpha}[\Phi_{\alpha} \circ F_{\alpha,r}^{-1}]\|_{\infty} \leq \mathbf{C} \|\varphi\|_{C^1} (r+1)$  and  $\|\Phi_{\alpha} \circ F_{\alpha,r}^{-1}\|_{\infty} \leq \mathbf{C} \|\varphi\|_{C^1} (r+1)$ . This is true because

$$(\Phi_{\alpha} \circ F_{\alpha,r}^{-1})(z) = \varphi\left(\frac{z_r+1}{2}\right) + \sum_{j=0}^{r-1} \varphi(T_{\alpha}^j z_r) = \varphi\left(\frac{z_r+1}{2}\right) + \sum_{j=1}^r \varphi(z_j),$$

and  $|\partial_{\alpha} z_r| \leq \mathbf{C}$  by Lemma 5.6.

Hence we have verified assumptions of Theorem 2.1 and Corollary 2.4 as required.

### 5.3 Proof of Theorem 1.2

Recall that the invariant measure of  $T_{\alpha}$  is denoted by  $\nu_{\alpha}$ , and its density by  $\rho_{\alpha}$ , while the invariant measure of the induced map  $F_{\alpha}$  is denoted by  $\mu_{\alpha}$ , and its density by  $h_{\alpha}$ .

**Lemma 5.9.**  $\rho_{\alpha}(z) = g_{\alpha}(z) / \int_0^1 g_{\alpha}(x) dx$  for all  $z \in (0, 1]$ , where

$$g_{\alpha}(z) = \frac{1}{2} \sum_{k=0}^{\infty} h_{\alpha}\left(\frac{z_k+1}{2}\right) z'_k.$$

*Proof.* Let  $\varphi$  be a nonnegative observable on  $[0, 1]$ , and  $\Phi_{\alpha} = \sum_{k=0}^{\tau_{\alpha}-1} \varphi \circ T_{\alpha}^k$  be the corresponding induced observable. In the beginning of Section 3 we partitioned the interval  $[1/2, 1]$  into intervals  $[y_{r+1}, y_r]$ ,  $r \geq 0$ , where  $F_{\alpha}$  has full branches and  $\tau_{\alpha} = r+1$ .

Compute

$$\begin{aligned}
\int \Phi_\alpha d\mu_\alpha &= \int_{1/2}^1 \sum_{k=0}^{\tau_\alpha(y)-1} \varphi(T_\alpha^k y) h_\alpha(y) dy = \sum_{j=0}^\infty \int_{y_{j+1}}^{y_j} \sum_{k=0}^j \varphi(T_\alpha^k y) h_\alpha(y) dy \\
&= \sum_{k=0}^\infty \sum_{j=k}^\infty \int_{y_{j+1}}^{y_j} \varphi(T_\alpha^k y) h_\alpha(y) dy = \sum_{k=0}^\infty \int_{1/2}^{y_k} \varphi(T_\alpha^k y) h_\alpha(y) dy \\
&= \int_{1/2}^1 \varphi(y) h_\alpha(y) dy + \frac{1}{2} \sum_{k=1}^\infty \int_0^{x_k} \varphi(T_\alpha^{k-1} x) h_\alpha\left(\frac{x+1}{2}\right) dx \\
&= \int_{1/2}^1 \varphi(y) h_\alpha(y) dy + \frac{1}{2} \sum_{k=0}^\infty \int_0^{x_{k+1}} \varphi(T_\alpha^k x) h_\alpha\left(\frac{x+1}{2}\right) dx \\
&= \int_{1/2}^1 \varphi(y) h_\alpha(y) dy + \frac{1}{2} \sum_{k=0}^\infty \int_0^{1/2} \varphi(z) h_\alpha\left(\frac{z_k+1}{2}\right) z'_k dz \\
&= \frac{1}{2} \sum_{k=0}^\infty \int_0^1 \varphi(z) h_\alpha\left(\frac{z_k+1}{2}\right) z'_k dz.
\end{aligned}$$

First we made a substitution  $x = T_\alpha y = 2y - 1$ , and then a substitution  $z = T_\alpha^k x$ , i.e.  $x = z_k$ . In the last step we used the fact that for  $z \geq 1/2$

$$h_\alpha(z) = (P_\alpha h_\alpha)(z) = \sum_{k=0}^\infty h_\alpha\left(\frac{z_k+1}{2}\right) \frac{z'_k}{2}.$$

Since  $\int \varphi d\nu_\alpha = \int \Phi_\alpha d\mu_\alpha / \int \tau_\alpha d\mu_\alpha$ , the result follows.  $\square$

**Lemma 5.10.**  $g_\alpha(z)$  and  $\partial_\alpha g_\alpha(z)$  are jointly continuous in  $\alpha, z$  on  $[\alpha_-, \alpha_+] \times (0, 1]$ . Also,  $0 \leq g_\alpha(z) \leq \mathbf{C} z^{-\alpha}$  and  $|\partial_\alpha g_\alpha(z)| \leq \mathbf{C} z^{-\alpha} (1 - \log z)^3$ .

*Proof.* By Theorem 2.1,  $\|h_\alpha\|_{C^2} \leq \mathbf{C}$  and  $\|\partial_\alpha h_\alpha\|_{C^1} \leq \mathbf{C}$ , and both  $h_\alpha(z)$  and  $\partial_\alpha h_\alpha(z)$  are jointly continuous in  $\alpha$  and  $z$ . By Lemma 5.3,  $0 \leq z'_r \leq \mathbf{C} (1 + rz^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha}$ , hence

$$0 \leq \sum_{r=1}^\infty z'_r \leq \mathbf{C} \int_1^\infty (1 + tz^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} dt \leq \mathbf{C} z^{-\alpha}. \quad (5.10)$$

Now,

$$0 \leq g_\alpha(z) = \frac{1}{2} \sum_{k=0}^\infty h_\alpha\left(\frac{z_k+1}{2}\right) z'_k \leq \mathbf{C} z^{-\alpha}.$$

Terms of the series are jointly continuous in  $\alpha$  and  $z$ , and convergence is uniform away from  $z = 0$ , thus  $g_\alpha(z)$  is also jointly continuous in  $\alpha$  and  $z$ .

Denote  $u_{\alpha,k}(z) = h_\alpha((z_k+1)/2) z'_k/2$ , so that  $g_\alpha(z) = \sum_{k=0}^\infty u_{\alpha,k}(z)$  and compute

$$\partial_\alpha u_{\alpha,k}(z) = \left[ (\partial_\alpha h_\alpha)\left(\frac{z_k+1}{2}\right) + h_\alpha\left(\frac{z_k+1}{2}\right) \frac{\partial_\alpha z_k}{2} \right] \frac{z'_k}{2} + h_\alpha\left(\frac{z_k+1}{2}\right) \frac{\partial_\alpha z'_k}{2}.$$

By Lemma 5.6,

$$0 \leq \partial_\alpha z_r \leq \mathbf{C} r^{-1/\alpha} \log(r z^\alpha) [\log r - \log z].$$

By Lemma 5.7,

$$|\partial_\alpha z'_r| \leq \mathbf{C} z'_r [\log(r z^\alpha)]^2 [\log r - \log z].$$

Thus  $|\partial_\alpha u_{\alpha,k}(z)| \leq \mathbf{C} z'_r [\log(r z^\alpha)]^2 [\log r - \log z]$ . Thus by Lemma 5.3,

$$\begin{aligned} \sum_{k=0}^{\infty} |\partial_\alpha u_{\alpha,k}(z)| &\leq \mathbf{C} \sum_{k=0}^{\infty} z'_k [\log(k z^\alpha)]^2 [\log k - \log z] \\ &\leq \mathbf{C} \int_1^{\infty} (1 + t z^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} [\log(t z^\alpha)]^2 [\log t - \log z] dt \\ &= \mathbf{C} z^{-\alpha} \int_{z^\alpha}^{\infty} (1 + s \alpha 2^\alpha)^{-(\alpha+1)/\alpha} (\log s)^2 \left[ \log \frac{s}{z^\alpha} - \log z \right] ds \\ &\leq \mathbf{C} z^{-\alpha} (1 - \log z). \end{aligned}$$

Therefore we can write

$$(\partial_\alpha g_\alpha)(z) = \sum_{k=0}^{\infty} \partial_\alpha u_{\alpha,k}(z).$$

Away from  $z = 0$ , the terms of the series are jointly continuous in  $\alpha$  and  $z$ , and series converges uniformly, so  $(\partial_\alpha g_\alpha)(z)$  is jointly continuous in  $\alpha$  and  $z$ , and  $|\partial_\alpha g_\alpha(z)| \leq \mathbf{C} z^{-\alpha} (1 - \log z)$ .  $\square$

**Corollary 5.11.**  $\rho_\alpha(z)$  and  $\partial_\alpha \rho_\alpha(z)$  are jointly continuous in  $\alpha$  and  $z$ . Also,  $0 \leq g_\alpha(z) \leq \mathbf{C} z^{-\alpha}$  and  $|\partial_\alpha g_\alpha(z)| \leq \mathbf{C} z^{-\alpha} (1 - \log z)$ .

*Proof.* Note that  $\int_0^1 g_\alpha(z) dz = \int \tau_\alpha d\mu_\alpha \geq 1$ , and

$$\frac{d}{d\alpha} \int_0^1 g_\alpha(z) dz = \int_0^1 (\partial_\alpha g_\alpha)(z) dz.$$

By Lemma 5.10,  $\int_0^1 g_\alpha(z) dz$  is continuously differentiable in  $\alpha$ , its derivative is bounded by  $\mathbf{C}$ . The result follows from Lemma 5.10 and relation, established in Lemma 5.9:

$$\rho_\alpha(z) = g_\alpha(z) / \int_0^1 g_\alpha(x) dx.$$

$\square$

**Corollary 5.12.** Assume that  $\varphi \in L^q[0, 1]$ , where  $q > (1 - \alpha_+)^{-1}$ . Then the map  $\alpha \mapsto \int \varphi(x) \rho_\alpha(x) dx$  is continuously differentiable on  $[\alpha_-, \alpha_+]$ .

*Proof.* Let  $p = 1/(1 - 1/q)$ . Then  $p < 1/\alpha_+$  and by Corollary 5.11,  $\|\partial_\alpha \rho_\alpha\|_{L^p}$  is bounded uniformly in  $\alpha$ . Since  $\rho_\alpha(x)$  and  $(\partial_\alpha \rho_\alpha)(x)$  are jointly continuous in  $\alpha$  and  $x$ , we can write

$$\left| \frac{d}{d\alpha} \int \varphi(x) \rho_\alpha(x) dx \right| = \left| \int_0^1 \varphi(x) (\partial_\alpha \rho_\alpha)(x) dx \right| \leq \|\varphi\|_{L^q} \|\partial_\alpha \rho_\alpha\|_{L^p}.$$

It is clear that the above is bounded on  $[\alpha_-, \alpha_+]$ . Continuity of  $\int_0^1 \varphi(x) (\partial_\alpha \rho_\alpha)(x) dx$  follows from continuity of  $(\partial_\alpha \rho_\alpha)(x)$  in  $\alpha$  and the dominated convergence theorem.  $\square$

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